THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 1

1. (a) Suppose X is separable. By definition, there exists a countable dense set D such that $\overline{D} = X$. Since D is countable, $X\backslash D$ is an open set. Hence D is a closed set and $\overline{D} = D$. However this leads to $D = \overline{D} = X$. Since X is uncountable, this leads to contradiction. Hence X is not separable.

   (b) Assume that the cocountable topology is C_I. Pick any $x \in X$. There exists a countable local base $\{U_n\}_{n\in\mathbb{N}}$ at x. By definition, $X\backslash U_n$ is countable. Hence $\cup_{n\in\mathbb{N}} X\backslash U_n = X\backslash \cap_{n\in\mathbb{N}} U_n$ is also countable. This shows that $\cap_{n\in\mathbb{N}} U_n$ must be an uncountable set.

   Pick any $z \in \cap_{n\in\mathbb{N}} U_n$ and $z \neq x$. Consider the open set $X\backslash\{z\}$. Clearly we have $x \in X\backslash\{z\}$. Furthermore, since $z \in \cap_{n\in\mathbb{N}} U_n$, we have $U_n \not\subset X\backslash\{z\}$ for all $n \in \mathbb{N}$. This contradicts with the fact that $\{U_n\}_{n\in\mathbb{N}}$ is a local base at x.

2. (a) To show that B is a base, we have to check:

   - For any $x \in X$, there exists some element $D \in B$ such that $x \in D$;
   - For any $U, V \in B$ and any $x \in U \cap V$, there exists $W \in B$ such that $x \in W \subset U \cap V$.

   It is clear that $x \in (x-1, x+1)$ for any $x \in \mathbb{R}$. Next, pick any $U, V \in B$ and $x \in U \cap V$.

   i. If $U = (a, b)$ and $V = (c, d)$ such that $a < c < b < d$, then we have
   $$x \in (c, b) \subset (a, b) \cap (c.d)$$
   ii. If $U = (a, b)$ and $V = (c, d)\backslash K$ such that $a < c < b < d$, then we have
   $$x \in (c, b)\backslash K \subset (a, b) \cap (c.d)\backslash K$$
   iii. If $U = (a, b)\backslash K$ and $V = (c, d)$ such that $a < c < b < d$, then we have
   $$x \in (c, b)\backslash K \subset (a, b)\backslash K \cap (c.d)$$
   iv. If $U = (a, b)\backslash K$ and $V = (c, d)\backslash K$ such that $a < c < b < d$, then we have
   $$x \in (c, b)\backslash K \subset (a, b)\backslash K \cap (c.d)\backslash K$$

   As a result, B is a base.

   (b) To show that $T_l \not\subset T_K$, note that the interval $[0, 1) \in T_l$ and $0 \in [0, 1)$. However, if I is an element in B containing 0, we have $I = (a, b)$ or $I = (a, b)\backslash K$ where $a < 0 < b$. In both cases, we have $\frac{a}{2} \in I$ and $I \not\subset [0, 1)$. So $[0, 1) \notin T_K$.

   To show that $T_K \not\subset T_l$, note that the interval $(-1, 1)\backslash K \in T_K$. However, if I is an elements in the base of lower limit topology containing 0, we have $I = [a, b)$ for some $a \leq 0 < b$. In particular, $\frac{1}{n} \in I$ for sufficiently large n and $I \not\subset (-1, 1)\backslash K$. So $(-1, 1)\backslash K \notin T_l$.

3. (a) ($\Rightarrow$) Let $x \in \overline{A} = A \cup A'$. If $x \in A$, then for any $U \in \mathfrak{T}$ with $x \in U$, we have $x \in U \cap A$ and hence $U \cap A \neq \emptyset$. If $x \in A'$ and $x \notin A$, by definition of A', for any $U \in \mathfrak{T}$ with $x \in U$, we have $U \cap A \backslash \{x\} \neq \emptyset$. Since $x \notin A$, we have $U \cap A = U \cap A \backslash \{x\} \neq \emptyset$.

   ($\Leftarrow$) Conversely, assume that for any $U \in \mathfrak{T}$ with $x \in U$, we have $U \cap A \neq \emptyset$. If $x \in A$, then we are done. If $x \notin A$, then for any $U \in \mathfrak{T}$ with $x \in U$, we have $U \cap A \backslash \{x\} = U \cap A \neq \emptyset$. Hence $x \in A'$.

   (b) It follows easily by (a).

   (c) Suppose A is open. Then $X \backslash A$ is closed. Hence

$$\overline{A} \backslash \operatorname{Frt}(A) = \overline{A} \backslash (\overline{A} \cap \overline{X \backslash A}) = \overline{A} \backslash (\overline{A} \cap (X \backslash A)) = (\overline{A} \backslash \overline{A}) \cup (\overline{A} \backslash (X \backslash A)) = \overline{A} \cap A = A$$

   Conversely, suppose $A = \overline{A} \backslash \operatorname{Frt}(A)$. Pick any $x \in A$. Since $x \notin \operatorname{Frt}(A)$, there exists $U \in \mathfrak{T}$ with $x \in U$ such that $U \cap A = \emptyset$ or $U \cap (X \backslash A) = \emptyset$. Since $X \in U \cap A$, we must have $U \cap (X \backslash A) = \emptyset$. This implies that $x \in U \subset A$. Hence A is open.

   (d) Suppose $x \in \operatorname{Int}(A)$. This implies that there exists $U \in \mathfrak{T}$ such that $x \in U \subset A$. In particular, we have $U \cap (X \backslash A) = \emptyset$. So $x \notin \operatorname{Frt}(A)$.

   (e) ($\Leftarrow$) Since A is closed, we have $\overline{A} = A$. Since A is open, $X \backslash A$ is closed and we have $\overline{X \backslash A} = X \backslash A$. Hence $\operatorname{Frt}(A) = \overline{A} \cap \overline{X \backslash A} = A \cap (X \backslash A) = \emptyset$.

   ($\Rightarrow$) Pick any $x \in A$. Since $x \notin \operatorname{Frt}(A)$, there exists $U \in \mathfrak{T}$ containing x such that $U \cap A = \emptyset$ or $U \cap (X \backslash A) = \emptyset$. Since $x \in U \cap A$, we must have $U \cap (X \backslash A) = \emptyset$. Hence we have $x \in U \subset A$. This shows that A is open. By (c), we have $A = \overline{A} \backslash \operatorname{Frt}(A) = \overline{A}$. Hence A is also closed.

   (f) Consider $A = [0,1] \cap \mathbb{Q}$. Note that $\overline{A} = [0,1]$, $\overline{\mathbb{R} \backslash A} = \mathbb{R}$. Hence $\operatorname{Frt}(A) = \overline{A} \cap \overline{\mathbb{R} \backslash A} = [0,1]$ and $\operatorname{Frt}(\operatorname{Frt}(A)) = \{0,1\}$. Clearly we have $\operatorname{Frt}(A) \neq \operatorname{Frt}(\operatorname{Frt}(A))$.